Activities

Activity 3: Various Problems

For the following problems fill in the gaps on the proof, they are highlighted in blue. Any comment in brown shall be discussed with your classmates and resolved as a group.

1. In this problem we will get to know one of the most important functions in mathematics: the *Riemann-Zeta function*. It is closely related to the distribution of prime numbers in the natural numbers, and one of the most challenging problems in mathematics, the Riemann Hypothesis, tries to describe it. Because I am not an expert on the topic I took the liberty of finding two links that would explain, better than me, how relevant this function is: Riemann-Zeta function, Riemann Hypothesis.

   For $1 < s < \infty$, define
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

   Prove that

   (a) $\zeta(s) = s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx$.

   (b) $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{x-[x]}{x^{s+1}}\,dx$.

   Here $[x]$ is defined to be the greatest integer m such that $m \leq x$.

   Proof.

   (a) We will start by writing the right hand side as a series:
$$s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx = s\sum_{n=1}^{\infty} n\int_n^{n+1} \frac{1}{x^{s+1}}\,dx.$$

   Compute the integral and conclude that it is equal to the Riemann-Zeta function. (You have to do this computation.)

   (b) Prove the equality.

   □

2. On another topic, we will try to study the relationship between the Riemann-Stieltjes integral and the Riemann integral.

   Suppose α increases monotonically on $[a, b]$, g is continuous, and $g(x) = G'(x)$ for $a \leq x \leq b$. Prove that
$$\int_a^b \alpha(x)g(x)\,dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\,d\alpha.$$

   Proof. Notice that
$$\sum_{i=1}^{n} \alpha(x_i)g(t_i)\Delta x_i = \sum_{i=1}^{n} \alpha(x_i)\left(G(x_i) - G(x_{i-1})\right)\Delta x_i.$$

   Try to express the latter expression as a discrete approximation of the result we are looking to prove (What are the conditions needed on G and α so that we can make that discrete approximation into a Riemann-Stieltjes integral?). Now, we know that α is non-decreasing, its only discontinuities are jumps, and for any $\epsilon > 0$ there can be only a finite number of jumps larger than ϵ (why?). Then, we can argue that any partition that is sufficiently fine will have upper and lower Riemann sums that differ by less than ϵ (make this more precise). Therefore $\alpha(x)g(x)$ is integrable and we are done. $\square$