Activities

Activity 4: Various Problems

For the following problems fill in the gaps on the proof, they are highlighted in blue. Any comment in brown shall be discussed with your classmates and resolved as a group.

1. Show that integration by parts can sometimes be applied to the "improper" integrals defined in Exercises 7 and 8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance, show that

$$\int_0^\infty \frac{\cos x}{1+x}\,dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx$$

Proof. It will be enough if we can show the following theorem

Theorem 1. *Let $f(x)$ and $g(x)$ be continuously differentiable functions defined on $[a,\infty)$ such that $\lim_{b\to\infty} f(b)g(b)$ exists and the integral $\int_a^\infty f(x)g'(x)\,dx$ converges. Then $\int_a^\infty f'(x)g(x)\,dx$ converges and*

$$\int_a^\infty f'(x)g(x)\,dx = \lim_{b\to\infty}[f(b)g(b) - f(a)g(a)] - \int_a^\infty f(x)g'(x)\,dx.$$

Discuss why the hypothesis should be sufficient, and prove the theorem. Using the theorem check the equality for the example. □

2. Let $\gamma_1, \gamma_2, \gamma_3$ be curves in the complex plane defined on $[0, 2\pi]$ by

$$\gamma_1(t) = e^{it}, \qquad \gamma_2(t) = e^{2it}, \qquad \gamma_3(t) = e^{2\pi i t \sin\left(\frac{1}{t}\right)}.$$

Show that these curves have the same range, that γ_1 and γ_2 are rectifiable, that the length of γ_1 is 2π, that the length of γ_2 is 4π, and that γ_3 is not rectifiable.

Proof. Notice that both γ_1, γ_2 and γ_3 have the same range, namely, their graph is the unit circle in the complex plane (check this). Now, for γ_1 and γ_2 the rectification is given by

$$l(\gamma_1) = \int_0^{2\pi} |\gamma_1'(t)|\,dt = 2\pi,$$
$$l(\gamma_2) = \int_0^{2\pi} |\gamma_2'(t)|\,dt = 4\pi.$$

Now, we show that γ_3 is not rectifiable. First notice that

$$\int_0^{2\pi} \left|\sin\left(\frac{1}{t}\right) - \frac{\cos\left(\frac{1}{t}\right)}{t}\right| dt \geq \int_0^{2\pi} \left|\frac{\cos\left(\frac{1}{t}\right)}{t}\right| dt - 2\pi$$

(why do you think we are trying to take this bound?). By using the substitution $u = \frac{1}{t}$ on the integral on the left, we see that it diverges (you have to show this). So we conclude that is not rectifiable (why?). □