Activities

Activity 1: Newton's Method

Suppose f is twice differentiable on $[a, b]$, $f(a) < 0, 0 < f(b)$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. Let ζ be the unique point in (a, b) at which $f(\zeta) = 0$ (why is ζ unique?).

Complete the details in the following outline of *Newton's method* for computing ζ.

1. Choose $x_1 \in (\zeta, b)$ and define $\{x_n\}$ by

   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

   Interpret this geometrically, in terms of a tangent of the graph f (May be drawing will be a good way to start. What would be a good choice of x_1?).

2. Prove that $x_{n+1} < x_n$ and that

   $$\lim_{n\to\infty} x_n = \zeta.$$

   After we have that $x_{n+1} < x_n$ we just need to show $f(x_{n+1}) > 0$ and that will give us the limit of the x_n converges to something bigger than ζ (why?). A uniqueness argument will give you the result.

   Proof. By induction we have $f(x_n) > 0$, this together with $f'(x_n) > 0$ we have by definition of x_{n+1} that $x_n > x_{n+1}$. Now, by the mean value theorem we have that there is $c \in (x_{n+1}, x_n)$ such that

   $$f'(c)(x_n - x_{n+1}) = f(x_n) - f(x_{n+1}).$$

   This, together with the fact that $f'(x_n) > f'(c)$ (why?) and $x_n > x_{n+1}$ gives us that $f(x_{n+1}) \geq 0$ (why?) and so because f is increasing $\zeta < x_{n+1} < x_n$. This tells you that x_n is a bounded decreasing sequence, which converges to some $\eta \geq \zeta$. Plugging η on the Newton's method equation we see that $f(\eta) = 0$ and by uniqueness of ζ, $\zeta = \eta$. $\square$

3. Use Taylor's theorem to show that

   $$x_{n+1} - \zeta = \frac{f''(t_n)}{2f'(x_n)}(x_n - \zeta)^2$$

   for some $t_n \in (\zeta, x_n)$.

   Proof. By Taylor's theorem we have that there exists a $t_n \in (\zeta, x_n)$ such that

   $$f(\zeta) = f(x_n) + f'(x_n)(\zeta - x_n) + \frac{f''(t_n)}{2}(\zeta - x_n)^2.$$

   Because $f(\zeta) = 0$ we have

   $$\begin{aligned} 0 &= f(x_n) + f'(x_n)(\zeta - x_n) + \frac{f''(t_n)}{2}(\zeta - x_n)^2 \\ -f(x_n) + f'(x_n)(x_n - \zeta) &= \frac{f''(t_n)}{2}(\zeta - x_n)^2 \\ \frac{-f(x_n)}{f'(x_n)} + (x_n - \zeta) &= \frac{f''(t_n)}{2f'(x_n)}(\zeta - x_n)^2 \\ x_{n+1} - x_n + (x_n - \zeta) &= \frac{f''(t_n)}{2f'(x_n)}(\zeta - x_n)^2 \\ x_{n+1} - \zeta &= \frac{f''(t_n)}{2f'(x_n)}(\zeta - x_n)^2. \end{aligned}$$

   $\square$

4. If $A = \frac{M}{2\delta}$, deduce that

$$0 \leq x_{n+1} - \zeta \leq \frac{1}{A}[A(x_1 - \zeta)]^{2^n}.$$

(Compare with exercise 3.16)

Proof. By the last item we have that

$$0 \leq x_{n+1} - \zeta \leq A(x_n - \zeta)^2$$
$$0 \leq x_2 - \zeta \leq A(x_1 - \zeta)^2 = \frac{1}{A}[A(x_1 - \zeta)]^2.$$

By induction we get what we wanted. (write it) □

5. Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

How does $g'(x)$ behave for x near ζ.?

Proof. We can see that $g(x) = x$ if and only if $f(x) = 0$. By computing the derivative of g we see that g goes to zero as $x \to \zeta$. This means that ζ is a critical point (why?). □

6. Put $f(x) = x^{1/3}$ on $(-\infty, \infty)$ and try Newton's method. What happens? (you should draw to understand the behavior around 0) Why did it fail?

Proof. The derivative is unbounded making the convergence of the Newton's method fail. More explicitly, if $x_n \neq 0$ we have $x_{n+1} = -2x_n$ (check). This tells you that $\limsup x_n = \infty$ and $\liminf x_n = -\infty$. □

Let us make a summary of what we just showed: With (a) and (b) we built a sequence of points that approach from the right to the unique zero of f on the interval $[a, b]$. With (c) and (d) we are able to get a bound on our error in the approximation. With (e) we see a reformulation to the problem of finding the zero of f. Finally, (f) shows you what happens with a function whose first derivative is not bounded around 0.