Activities

Activity 2: Existence and Uniqueness of Solutions to O.D.E. I: Uniqueness

For the following problems fill in the gaps on the proof, they are highlighted in blue. Any comment in brown shall be discussed with your classmates and resolved as a group. Let ϕ be a real function defined on a rectangle R in the plane, given by $a \leq x \leq b$, $\alpha \leq y \leq \beta$. A *solution* of the initial value problem

$$y' = \phi(x, y) \qquad\qquad y(a) = c$$

for $\alpha \leq c \leq \beta$, is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c$, $\alpha \leq f(x) \leq \beta$, and

$$f'(x) = \phi\left(x, f(x)\right)$$

for every $x \in [a, b]$.

1. Prove that such a problem has at most one solution if there is a constant A such that

   $$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

   whenever $(x, y_1) \in \mathbb{R}$ and $(x, y_2) \in \mathbb{R}$.

   Proof. Let f_1 and f_2 be two solutions of the system and consider $f = f_2 - f_1$. By hypothesis we know that

   $$|f'(x)| \leq A|f(x)|.$$

   We know $f(a) = f_2(a) - f_1(a) = 0$. By Problem 26 we know $f(x) = 0$ for $x \in [a, b]$. Discuss how this has showed, with the given hypothesis, the uniqueness of the solution. □

2. Check that the latter result does not work for the initial-value problem

   $$y' = y^{\frac{1}{2}}, \qquad\qquad y(0) = 0.$$

   Two distinct solutions could be $f(x) = 0$ and $f(x) = \frac{x^2}{4}$. Can you find all the other solutions?

   Proof. Make sure you understand why this differential equation has more than one solution, furthermore try and think what are the differences with respect to the previous problem. Consider f to be a solution so that $f(a) = 0$ and $f(x) > 0$ for $x \in (a, b)$. Why does a solution like this work out?. Consider the function $g(x) = \sqrt{f(x)}$. Notice that $g'(x) = \frac{1}{2}$, so for some constant c we have $g(x) = \frac{x+c}{2}$. Therefore,

   $$f(x) = \begin{cases} \frac{(x-a)^2}{4} & a \leq x \\ 0 & 0 \leq x \leq a. \end{cases}$$

   □

3. Consider the system of differential equations of the form

   $$y_j' = \phi_j(x, y_1, \dots, y_k), \qquad\qquad y_j(a) = c_j$$

   for $j \in \{1, \dots, k\}$. Formulate and prove an analogous uniqueness theorem for systems of differential equations like the one stated above.

Proof. This problem is exactly the same as for number one, you only need to get a similar result to that of Problem 26 for vector valued functions, i.e. functions $f : (a, b) \to \mathbb{R}^n$. After this, you can write the same proof word by word. $\square$

4. Consider the system

$$y_j' = y_{j+1}$$
$$y_k' = f(x) - \sum_{j=1}^{k} g_j(x) y_j,$$

where $f, g_1, \ldots, g_k$ are continuous real functions on $[a, b]$. Derive a uniqueness theorem for solutions of the equation

$$y^{(k)} + g_k(x) y^{(k-1)} + \cdots + g_2(x) y' + g_1(x) y = f(x),$$

subject to initial conditions

$$y(a) = c_1, \qquad y'(a) = c_2, \qquad \cdots, \qquad y^{(k-1)}(a) = c_k.$$

Proof. Let $\mathbf{y} = (y_1, y_2, \ldots, y_k) = \left(y, y', \ldots, y^{(k-1)}\right)$ and

$$\phi(x, \mathbf{y}) = \left(y_2, \ldots, y_k, f(x) - \sum_{j=1}^{k} g_j(x) y_j\right).$$

Notice that if we have to vectors $\mathbf{u}, w$ then

$$|\phi(x, \mathbf{w}) - \phi(x, \mathbf{u})| = \left|\left(\mathbf{w}_2 - \mathbf{u}_2, \ldots, \mathbf{w}_k - \mathbf{u}_k, f(x) - \sum_{j=1}^{k} g_j(x) \left(\mathbf{u}_j - \mathbf{w}_j\right)\right)\right|.$$

We can find M(what should M be?) such that

$$|\phi(x, \mathbf{w}) - \phi(x, \mathbf{u})| \leq (M+1) \sum_{j=1}^{k} |\mathbf{u}_j - \mathbf{w}_j| \leq k (M+1) |\mathbf{w} - \mathbf{u}|.$$

Discuss why this would be enough to be your hypothesis for the uniqueness theorem, and explain why is there a unique solution to the initial value problem. $\square$