Activities

Activity 1: Newton's Method

Suppose f is twice differentiable on $[a, b]$, $f(a) < 0, 0 < f(b)$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. Let ζ be the unique point in (a, b) at which $f(\zeta) = 0$ (why is ζ unique?).

Complete the details in the following outline of *Newton's method* for computing ζ.

1. Choose $x_1 \in (\zeta, b)$ and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

   Interpret this geometrically, in terms of a tangent of the graph f (May be drawing will be a good way to start. What would be a good choice of x_1?).

2. Prove that $x_{n+1} < x_n$ and that

$$\lim_{n\to\infty} x_n = \zeta.$$

   After we have that $x_{n+1} < x_n$ we just need to show $f(x_{n+1}) > 0$ and that will give us the limit of the x_n converges to something bigger than ζ (why?). A uniqueness argument will give you the result.

3. Use Taylor's theorem to show that

$$x_{n+1} - \zeta = \frac{f''(t_n)}{2f'(x_n)}(x_n - \zeta)^2$$

   for some $t_n \in (\zeta, x_n)$.

4. If $A = \frac{M}{2\delta}$, deduce that

$$0 \leq x_{n+1} - \zeta \leq \frac{1}{A}[A(x_1 - \zeta)]^{2^n}.$$

   (Compare with exercise 3.16)

5. Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

   How does $g'(x)$ behave for x near ζ.?

6. Put $f(x) = x^{1/3}$ on $(-\infty, \infty)$ and try Newton's method. What happens? (you should draw to understand the behavior around 0) Why did it fail?

Let us make a summary of what we just showed: With (a) and (b) we built a sequence of points that approach from the right to the unique zero of f on the interval $[a, b]$. With (c) and (d) we are able to get a bound on our error in the approximation. With (e) we see a reformulation to the problem of finding the zero of f. Finally, (f) shows you what happens with a function whose first derivative is not bounded around 0.