Outlines

Homework 1

1. (5.2) Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)}.$$

Proof. To show that f is increasing take $a < s < t < b$ and use the mean value theorem on (s, t) then

$$f(t) - f(s) = f'(p)(t - c) > 0$$

where $p \in (s, t)$. The inverse g of f is continuous. Notice that if $f(x) = y$ and $f(u) = k$ with $x, u \in (a, b)$

$$\frac{g(y) - g(k)}{y - k} - \frac{1}{f'(x)} = \frac{x - u}{f(x) - f(u)} - \frac{1}{f'(x)} = \frac{1}{\frac{f(x)-f(u)}{x-u}} - \frac{1}{f'(x)}.$$

Because the derivative is positive for all elements in (a, b) we have that

$$\lim_{u \to x} \frac{1}{\frac{f(x)-f(u)}{x-u}} = \frac{1}{\lim_{u \to x} \frac{f(x)-f(u)}{x-u}}$$

so that for any $\epsilon > 0$ there is $\eta > 0$ such that

$$\left| \frac{g(y) - g(k)}{y - k} - \frac{1}{f'(x)} \right| < \epsilon$$

for any u with $|u - x| < \eta$. Since $|u - x| = |g(y) - g(k)|$ there is a $\delta > 0$ such that if for k with $|y - k| < \delta$ we have

$$\left| \frac{g(y) - g(k)}{y - k} - \frac{1}{f'(x)} \right| < \epsilon$$

. □

2. (5.4) If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

where $C_0, \ldots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof. Consider

$$f(x) = C_0 x + \frac{C_1 x^2}{2} + \cdots + \frac{C_{n-1} x^n}{n} + \frac{C_n x^{n+1}}{n+1}.$$

By the mean value theorem you have that there is a $c \in [0, 1]$ such that $f'(c) = 0$ (you have to check the hypothesis and check that the right hand side is indeed 0). □

3. (5.6) Suppose

(a) f is continuous for $x \geq 0$,

(b) $f'(x)$ exists for $x > 0$,

(c) $f(0) = 0$,

(d) f' is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x}$$

and prove that g is monotonically increasing.

Proof. By (a) and (b) we have that f is continuous on $[0, t]$ and differentiable on $(0, t)$. Apply the mean value theorem to f on that interval. This will tell you by (c) and (d) that the derivative of g is positive (why?), and so it is monotonically increasing. □

4. (5.8) Suppose f' is continuous on $[a, b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

whenever $0 < |t - x| < \delta, a \leq x \leq b, a \leq t \leq b$. Does this hold for vector-valued functions too?

Proof. This is a consequence from the fact that f' is uniformly continuous (it is continuous on a compact set) and the mean value theorem. Let $\epsilon > 0$ and let $\delta > 0$ be the δ you get from uniform continuity of f'. Let $t, x \in [a, b]$ such that $0 < |t - x| < \delta$, by the mean value theorem there is $u \in (t, x)$ (without loss of generality assume $x < t$) such that

$$\frac{f(t) - f(x)}{t - x} = f'(u).$$

We know $|u - x| < |t - x| < \delta$ so by uniform continuity

$$|f'(u) - f'(x)| = \left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon.$$

□

5. (5.9) Let f be a continuous real function on $\mathbb{R}^1$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?

Proof. Yes, by L'Hopital's rule (Check). □

6. (5.14) Let f be a differentiable real function defined in (a, b). Prove that f is convex if and only if f' is monotonically increasing. Assume next that $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.

Proof. Let $x < y$ in (a, b). You can show that the inequality stating that f is convex on (x, y) can be rewritten (prove this) as

$$\frac{f(y) - f(z)}{y - z} \geq \frac{f(z) - f(x)}{z - x}$$

for $x < z < y$. By the mean value theorem on (x, z) and (z, y), and because f' is increasing the latter inequality holds (why was this enough?). For the other direction you can use that f convex on $[a, b]$ implies

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(t)}{u - t} \leq \frac{f(v) - f(u)}{v - u}$$

for every $a < s < t < u < v < b$. Taking the limit as $s \to t$ and the limit as $v \to u$ you get that f' is increasing. □

7. (5.19) Suppose f is defined in $(-1,1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1, \alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$. Define the difference quotients

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}.$$

Prove the following statements:

(a) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$.

(b) If $0 < \alpha_n < \beta_n$ and $\beta_n/(\beta_n - \alpha_n)$ is bounded, then $\lim D_n = f'(0)$.

(c) If f' is continuous in $(-1,1)$, then $\lim D_n = f'(0)$.

Give an example in which f is differentiable in $(-1,1)$ and in which α_n, β_n tends to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.

Proof. (a) Show that

$$D_n = \frac{\beta_n}{\beta_n - \alpha_n}\frac{f(\beta_n) - f(0)}{\beta_n} + \frac{-\alpha_n}{\beta_n - \alpha_n}\frac{f(\alpha_n) - f(0)}{\alpha_n}.$$

Let $\epsilon > 0$, then because the derivative exists around 0 there is a $\delta > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$ (why?)

$$\left|\frac{f(\beta_n) - f(0)}{\beta_n} - f'(0)\right| < \epsilon \qquad \left|\frac{f(\alpha_n) - f(0)}{\alpha_n} - f'(0)\right| < \epsilon.$$

Now, for every $n/geqN$

$$|D_n - f'(0)| \leq \frac{\epsilon\beta_n}{\beta_n - \alpha_n} - \frac{\alpha_n\epsilon}{\beta_n - \alpha_n} = \epsilon.$$

(why was the inequality true?)

(b) Let M be the bound for $\frac{\beta_n}{\beta_n - \alpha_n}$. Notice this bound also bounds $\frac{\alpha_n}{\beta_n - \alpha_n}$. Let $\epsilon > 0$ and as before choose $N \in \mathbb{N}$ so that for $n \geq N$

$$\left|\frac{f(\beta_n) - f(0)}{\beta_n} - f'(0)\right| < \frac{\epsilon}{2M} \qquad \left|\frac{f(\alpha_n) - f(0)}{\alpha_n} - f'(0)\right| < \frac{\epsilon}{2M}.$$

Similarly to the last item, for every $n/geqN$

$$|D_n - f'(0)| \leq \frac{\epsilon\beta_n}{2M(\beta_n - \alpha_n)} + \frac{\alpha_n\epsilon}{2M(\beta_n - \alpha_n)} < \epsilon.$$

(why was the inequality true?)

(c) Because f' is continuous we can use the mean value theorem and find $\gamma_n \in (\alpha_n, \beta_n)$ so that $D_n = f'(\gamma_n)$. Because $\gamma_n \to 0$ and f' is continuous, then $f'(\gamma_n) \to f'(0)$.

Think of the second example of 5.6. □

8. (5.26) Suppose f is differentiable on $[a,b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a,b]$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

Proof. Let $x_0 = a + \frac{1}{k_0 A}$, where k is a positive integer that allows you to have $x_0 < b$, and let M_0 and M_1 as in the hint of the problem. Then

$$|f(x)| \leq M_1|x - a| \leq M_0 A|x_0 - a| = \frac{M_0}{k}$$

for any $x \in (a, x_0)$. This implies that $M_0 \leq \frac{1}{k}M_0$, so $M_0 = 0$. We now do the same for $x_1 = x_0 + \frac{1}{kA}$. In a finite number of steps, $b < x_n + \frac{1}{k_n A}$ and we will be done. □

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