Outlines

Homework 2

1. (5.11): Suppose f is defined in a neighborhood of x, and suppose $f'(x)$ exists. Show that
$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=f''(x).$$
Show that the limit may exist even if $f''(x)$ does not.

Proof. By L'Hopital's rule, we have
$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}&=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}\\&=\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{2h}+\lim_{h\to 0}\frac{f'(x-h)-f'(x)}{2h}\\&=f''(x).\end{aligned}$$
□

2. (5.15): Suppose $a\in\mathbb{R}^1$, f is twice-differentiable real function on (a,∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, on (a,∞). Prove that
$$M_1^2\le 4M_0M_2.$$
Does the inequality hold for vector-valued functions too?

Proof. For the case where $M_0=M_2=+\infty$ the inequality holds. Assume that both M_0 and M_2 are finite. What we want to show is
$$|f'(x)|\le 2\sqrt{M_0M_2}$$
for all $x>a$. We can further assume that $M_2\neq 0$, since in that case f is a linear function which is bounded only when f is a constant. Hence, we can assume that $0<M_0, M_2\infty$. Following the hint we need to choose $h=\sqrt{\frac{M_0}{M_2}}$, and we obtain
$$|f'(x)|\le 2\sqrt{M_0M_2},$$
which is what we want. For equality you just need to follow the hint. The result also holds for vector valued functions (Here you just need to define a real valued function that depends on f, may be the dot product with the unitary vector defined by $u=\frac{f'(x_0)}{|f'(x_0)|}$ where $x_0>a$ where $a<M_1$.) □

3. (5.17): Suppose f is a real, three times differentiable function on $[-1,1]$, such that
$$f(-1)=0,\qquad f(0)=0,\qquad f(1)=1,\qquad f'(0)=0.$$
Prove that $f^{(3)}(x)\ge 3$ for some $x\in(-1,1)$.

Proof. Use Taylor's formula for -1 and 1, then there are $s,t\in(-1,1)$ such that
$$f(1)-f(-1)=\frac{f^{(3)}(s)+f^{(3)}(t)}{6}$$
(Check why the last equality is true) which implies $f^{(3)}(s)+f^{(3)}(t)=6$. □

4. (5.22): Suppose f is a real function on $(-\infty, \infty)$. Call x a *fixed point* if $f(x) = x$.

   (a) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

   (b) Show that the function f defined by

   $$f(t) = t + \frac{1}{1+e^t}$$

   has no fixed point, although $0 < f'(t) < 1$ for all real t.

   (c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and $x_{n+1} = f(x_n)$ for every $n \geq 1$.

   (d) Show that the process described in (c) can be visualized by the zig-zag path

   $$(x_1, x_2) \to \cdots \to (x_3, x_4) \to$$

   Proof.

   (a) Suppose that there are two fixed points $x < y$ and $f'(t) \neq 1$ for all t. By the Mean Value Theorem on (x, y) we have that for some $k \in (x, y)$

   $$y - x = f(y) - f(x) = f'(k)(x - y)$$

   which leads to $f'(k) = 1$, a contradiction. We conclude there is only one fixed point.

   (b) Try to set up $f(t) = t$, you arrive to a contradiction (why?). By computing the derivative we see that $f'(x) \neq 1$ for all x in the domain.(Does this contradict the last item?)

   (c) Choose $\{x_n\}$ as established in the problem, we will show that the sequence is a Cauchy sequence: we know

   $$|x_n - x_m| \leq \sum_{i=m}^{n} |x_i - x_{i-1}|$$

   for every $n > m > N$ for some N. Furthermore, because f' is bounded, by the Mean Value Theorem applied to (x_{i-1}, x_i) we have

   $$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq A|x_n - x_{n-1}| \leq A^{n-1}|x_2 - x_1|$$

   for $n \geq 1$. Therefore,

   $$\begin{aligned} |x_n - x_m| &\leq |x_2 - x_1| \left(\sum A^{i-2} \right) \\ &< \frac{|x_2 - x_1| A^{m-1}}{1 - A} \\ &\leq \frac{|x_2 - x_1| A^{N}}{1 - A}. \end{aligned}$$

   As N goes to infinity this quantity goes to 0 and so $\{x_n\}$ is Cauchy and it converges to some x. This x is a fixed point by continuity of f (check this).

   (d) Draw the sequence in the plane.

   □

5. (6.1): Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and that $\int f \, d\alpha = 0$.

Proof. We know this function is always 0, except at the point x_0. Let $\epsilon > 0$, we know there exists δ such that

$$|x - x_0| < \delta \Rightarrow |\alpha(x) - \alpha(x_0)| < \frac{\epsilon}{2}.$$

Now, let $a = t_0 \leq t_1 \leq \cdots t_n = b$ be a partition of $[a, b]$. We know that there is an i such that $t_{i-1} \leq x_0 \leq t_{i+1}$ (it could be that $x_0 = t_i$ so it is at most at two elements of the partition). Now, notice that for any $a_i \in [t_{i-1}, t_i]$ we have that

$$\begin{aligned}\left|\sum f(a_i)\left(\alpha(t_i) - \alpha(t_{i-1})\right)\right| &\leq f(a_{i+1})\left(\alpha(t_{i+1}) - \alpha(t_i)\right) + f(a_i)\left(\alpha(t_i) - \alpha(t_{i-1})\right) \\ &\leq \alpha(t_{i+1}) - \alpha(t_{i-1}) \\ &\leq \epsilon.\end{aligned}$$

Because this is true for arbitrary a_i, we have that $f \in \mathcal{R}$ and $\int f \, d\alpha = 0$. Why do we get this? □

6. (6.2): Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x) \, dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Proof. Suppose that there is $x_0 \in [a, b]$ such that $f(x_0) \neq 0$. Because $f(x)$ is continuous on $[a, b]$ and $\frac{f(x_0)}{2} > 0$, we know there is a $\delta > 0$ for which

$$|f(x_0) - f(x)| < \frac{f(x_0)}{2}$$

for all $x \in [a, b]$ with $|x - x_0| < \delta$. Now, consider $\eta = \min(x_0 - a, b - x_0, \delta)$ and let

$$I = \begin{cases} [x_0 - \eta, x] & \eta \in [a, b], \\ [x_0, x_0 + \eta] & \text{otherwise.} \end{cases}$$

Why do you think we chose η in this way? By construction, $I \subset [a, b]$ and

$$f(x_0) + (f(x) - f(x_0)) \geq f(x) - |f(x) - f(x_0)| > \frac{f(x_0)}{2}$$

for all $x \in I$. Define

$$f_1(x) = \begin{cases} f(x), & x \in I, \\ 0, & x \notin I. \end{cases} \qquad f_2(x) = \begin{cases} f(x), & x \notin I, \\ 0, & x \in I. \end{cases}$$

These two function are non-negative, bounded and continuous possibly at every point (where do you think it could be discontinuous?), which means that they are both Riemann-integrable. We know

$$\int_a^b f_1(x) \, dx \geq \eta \frac{\epsilon}{2}$$

and

$$\int_a^b f_2(x) \, dx \geq 0.$$

So that,

$$\int_a^b f(x) \, dx > 0$$

leading to a contradiction. To see the last inequality you might want to express f as a sum of the two functions we constructed. □

7. (6.3): Define three functions $\beta_1, \beta_2, \beta_3$ as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for all j; and $\beta_1(0) = 0, \beta_2(0) = 1, \beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.

   (a) Prove that $f \in \mathcal{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then

$$\int f \, d\beta_1 = f(0).$$

(b) State and prove a similar result for β_2.

(c) Prove that $f \in \mathcal{R}(\beta_3)$ if and only if f is continuous at 0.

(d) If f is continuous at 0 prove that

$$\int f\, d\beta_1 = \int f\, d\beta_2 = \int f\, d\beta_3 = f(0).$$

Proof. Let $t_0 < t_1 < \cdots < t_n$ be a partition of any interval containing 0, and without loss of generality assume that $t_k = 0$ for some $k \leq n$ (why can I do this?). With this our Riemann-Stieltjes sums (upper and lower) are: for $j = 1$, M_k and m_k; for $j = 2$ they will be M_{k-1}, and m_{k-1}; and for $j = 3$ they will be $\frac{M_{k-1}+M_{k-2}}{2}$ and $\frac{m_{k-1}+m_k}{2}$, respectively.

(a) By definition we have that $m_k \leq f(x) \leq M_k$ for every $x \in [0, t_{k+1}]$. That means that the sets of upper and lower sums are arbitrarily near to each other if and only if $M_k - m_k < \epsilon$. If such partition exists let $\delta = t_{k+1}$. Then

$$|f(x) - f(0)| \leq M_k - m_k < \epsilon$$

for $x \in (0, \delta)$, which implies

$$\lim_{x \to 0+} f(x) = f(0).$$

Conversely, if $\lim_{x\to 0+} f(x) = f(0)$ then we know there is a $\delta > 0$ such that $|f(x) - f(0)| < \epsilon$ for every $|x| < \delta$. Take a partition where $t_k = 0$ and $t_{k+1} = \delta$. Notice that by what we just showed we proved that the Riemann sums differ from $f(0)$ by less than ϵ (why?) and so

$$\int f\, d\beta_1 = f(0).$$

(b) Notice $f \in \mathcal{R}(\beta_2)$ if and only if $\lim_{x\to 0-} f(x) = f(0)$. If this holds, then $\int f d\beta_2 = f(0)$. The proof is almost identical to the one in the last item.

(c) In this case our Riemann-Stieltjes sum differ by

$$\frac{(M_k - m_k) + (M_{k-1} - m_{k-1})}{2}.$$

Let $\epsilon > 0$, there is a partition containing 0 for which this difference is less than $\frac{\epsilon}{2}$. Let $\delta = \min(t_{k+1} - t_{k-1})$. Then, for $x \in (-\delta, \delta)$ we have

$$\begin{aligned} |f(x) - f(0)| &\leq \max\left(\frac{M_k - m_k}{2}, \frac{M_{k-1} - m_{k-1}}{2}\right) \\ &\leq M_k - m_k + M_{k-1} - m_{k-1} < \epsilon, \end{aligned}$$

showing that f is continuous at 0. As before,

$$\int f\, d\beta_3 = f(0).$$

(d) It follows from the last three items.

□

8. (6.4): If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.

Proof. Every upper Riemann sum equals $b - a$, and every lower Riemann sum equals 0. why is this enough? □

Disclaimer: Some of these solutions have been taken from some outside sources, which will not be cited so that students do not find them. If you are interested in knowing where I got the solutions from please e-mail me.