Activities

Activity 5: Some examples

For the following problems fill in the gaps on the proof, they are highlighted in blue. Any comment in brown shall be discussed with your classmates and resolved as a group.

1. Let

$$f_n(x) = \begin{cases} 0 & \left(x < \frac{1}{n+1}\right), \\ \sin^2\left(\frac{\pi}{x}\right) & \left(\frac{1}{n+1} \leq x \leq \frac{1}{n}\right) \\ 0 & \left(x > \frac{1}{n}\right). \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

Proof. Start by drawing the first 2 or 3 functions in the sequence, and check what the pointwise limit look like for a given x. Define f in this manner and prove that f is the pointwise limit of your f_n's. Notice that this cannot be uniform convergence since for any n that you take, there will be a point x such that $f_n(x)=1$ (find such x).

Find what $\sum f_n$ converges to. Notice that since the terms are nonnegative the series converges absolutely. But, since the sum is not continuous at 0, the series does not converge uniformly on any interval containing 0 (why is this?). □

2. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Proof. We can write the series as

$$x^2 \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} + \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$$

Use the Weierstrass M-test to show that the first series converges both uniformly and absolutely on any bounded interval. The second converges uniformly in x and by problem 2 the sum of the two series converges uniformly. Justify why the sum of the series does not converge absolutely. □