Activities

Activity 6: Nowhere differentiable functions and other

For the following problems fill in the gaps on the proof, they are highlighted in blue. Any comment in brown shall be discussed with your classmates and resolved as a group.

1. Let E_n be the set of all $f \in C([0,1])$ for which there exists $x_0 \in [0,1]$ (depending on f) such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0,1]$. If we know that E_n is a nowhere dense set in $C([0,1])$ show that the set of nowhere differentiable functions is residual in $C([0,1])$.

   Here *residual* means that it is the complement of a set A that is a countable union of nowhere dense sets.

   Proof. Let us call $\mathcal{N}$ the set of continuous nowhere differentiable functions in $[0,1]$. Because each E_n is nowhere dense, if we are able to show that $\mathcal{N}^c \subset \bigcup E_n$ we would be done (think about this and discuss with your classmates about it). Let $f \in \mathcal{N}^c$, then there exists $x_0 \in [0,1]$ such that $f'(x_0)$ exists. Hence, there is $\alpha > 0$ such that for any $x \in [0,1]$ with $|x - x_0| < \alpha$ we have

   $$|f(x) - f(x_0)| \leq |x - x_0| \left(1 + |f'(x_0)|\right).$$

   We can also show that for $x \in [0,1]$ with $|x_0 - x| \geq \alpha$

   $$\left|\frac{f(x) - f(x_0)}{x - x_0}\right| \leq \frac{2\left(\sup_{x\in[0,1]} |f(x)|\right)}{\alpha}.$$

   Show that there is an n for which

   $$|f(x) - f(x_0)| \leq n|x - x_0|$$

   for all $x \in [0,1]$. For this n we have that $f \in E_n$, and so $\mathcal{N}^c \subset \bigcup E_n$. □

2. Suppose $\{f_n\}, \{g_n\}$ are defined on E, and

   (a) $\sum f_n$ has uniformly bounded partial sums;

   (b) $g_n \to 0$ uniformly on E;

   (c) $g_1 \geq g_2 \geq \cdots$ for every $x \in E$.

   Prove that $\sum f_n g_n$ converges uniformly on E. *Hint:* Compare with Theorem 3.42.