Activities

Activity: Existence and Uniqueness of Solutions to O.D.E.'s II: Existence.

For the following problems fill in the gaps on the proof, they are highlighted in blue. Any comment in brown shall be discussed with your classmates and resolved as a group.

1. Suppose ϕ is a continuous bounded real function in the strip defined by $0 \leq x \leq 1, -\infty < y < \infty$. Prove that the initial-valued problem

$$y' = \phi(x, y), \qquad y(0) = c$$

has a solution.

Compare the hypothesis of this problem with the hypothesis of Activity 2.

Hint: Fix n. For $i = 0, \cdots, n$, put $x_i = \frac{i}{n}$. Let f_n be a continuous function on $[0, 1]$ such that $f_n(0) = c$,

$$f_n'(t) = \phi\left(x_i, f_n(x_i)\right); \qquad x_i < t < x_{i+1},$$

and put

$$\Delta_n(t) = f_n'(t) - \phi\left(t, f_n(t)\right),$$

except at the points x_i, where $\Delta_n(t) = 0$. Then

$$f_n(x) = c + \int_0^x \left[\phi\left(t, f_n(t)\right) + \Delta_n(t)\right].$$

Choose $M < \infty$ so that $|\phi| \leq M$. Verify the following assertions.

(a) $|f_n'| \leq M, |\Delta_n| \leq 2M, \Delta_n \in \mathcal{R}$, and $|f_n| \leq |c| + M = M_1$, say, on $[0, 1]$, for all n.

(b) $\{f_n\}$ is equicontinuous on $[0, 1]$, since $|f_n'| \leq M$.

(c) Some $\{f_{n_k}\}$ converges to some f, uniformly on $[0, 1]$.

(d) Since ϕ is uniformly continuous on the rectangle $0 \leq x \leq 1, |y| \leq M_1$,

$$\phi\left(t, f_{n_k}(t)\right) \to \phi\left(t, f(t)\right)$$

uniformly on $[0, 1]$.

(e) $\Delta_n(t) \to 0$ uniformly on $[0, 1]$, since

$$\Delta_n(t) = \phi\left(x_i, f_n(x_i)\right) - \phi\left(t, f_n(t)\right)$$

in (x_i, x_{i+1}).

(f) Hence

$$f(x) = c + \int_0^x \phi\left(t, f(t)\right)\, dt.$$

This f is a solution of the given problem.

Proof. For a more general result we will assume that ϕ is a bounded continuous mapping from $[0,1] \times \mathbb{R}^k$ into $\mathbb{R}^k$ and that $c \in \mathbb{R}^k$. Notice that under this assumptions we will have the necessary pieces to get the more general result stated in the next problem. Define $f_n(t) = c + t\phi(0,c)$ for $0 \leq t \leq x_1$, and then, by induction on i,

$$f_n(t) = f_n(x_i) + (t - x)\phi(x_i, f_n(x_i))$$

for $x_i < t \leq x_{i+1}$. Confirm with your classmates that each f_n is a well defined function. By taking Δ_n as above, we get that

$$f_n'(t) = \Delta_n(t) + \phi(t, f_n(t))$$

except at a finite number of points, and therefore

$$f_n(x) = f_n(0) + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)]\, dt.$$

(a) Prove what was asked for in the hint.

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(e) For each t and n let $i(n)$ be chosen so that $t \in [x_{i(n)}, x_{i(n)+1}]$, so that $|t - x_{i(n)}| \leq \frac{1}{n}$. Show that this, together with $f_{n_k} \to f$ uniformly implies

$$\phi\left(x_{i(n)}, f_n(x_{i(n)})\right) - \phi(t, f_n(t)) \to 0.$$

(f) Prove what was asked for in the hint.

□

2. Prove an analogous existence theorem for the initial-value problem

$$\mathbf{y}' = \Phi(x, \mathbf{y}), \qquad \mathbf{y}(0) = \mathbf{c},$$

where now $\mathbf{c} \in \mathbb{R}^k$, $\mathbf{y} \in \mathbb{R}^k$ and Φ is a continuous bounded mapping of the part of $\mathbb{R}^{k+1}$ defined by $0 \leq x \leq 1, \mathbf{y} \in \mathbb{R}^k$ into $\mathbb{R}^k$.