Outlines

Homework 3

1. (6.5) Suppose f is a bounded real function on $[a, b]$ and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume that $f^3 \in \mathcal{R}$?

   Proof. Consider
   $$f_1 = \begin{cases} 1, & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$$
   This function is not in $\mathcal{R}$ but f^2 is (check this). By Theorem 6.11 it is true that $f^3 \in \mathcal{R}$, why didn't the same argument hold for the last part of the problem?. □

2. (6.7) Suppose f is a real function on $[0, 1]$ and $f \in \mathcal{R}$ on $[c, 1]$ for every $c > 0$. Define
   $$\int_0^1 f(x)\, dx = \lim_{c \to 0+} \int_0^1 f(x)\, dx$$
   if this limit exists (and is finite).

   (a) If $f \in \mathcal{R}$ on $[0, 1]$ show that this definition of the integral agrees with the old one.

   (b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

   Proof. (a) Suppose $f \in \mathcal{R}$ on $[0, 1]$. Let $\epsilon > 0$ be given, and let $M = \sup\{|f(x)| : 0 \leq x \leq 1\}$. Let $c \in \left(0, \frac{\epsilon}{4M}\right]$ be fixed, and consider any partition P of $[0, 1]$ containing c for which the upper and lower sums $\sum M_j \Delta t_j$ and $\sum m_j \Delta t_j$ of f differ by less than $\frac{\epsilon}{4}$. Taking the partition of $[c, 1]$ formed by the elements in the partition P that lie in $[c, 1]$ we have a partition for which the difference of the upper and lower sums differ by $\frac{\epsilon}{4}$. We just prove that for $c < \frac{\epsilon}{4}$ and a suitable partition containing c where
   $$\sum M_j \Delta t_j - \frac{\epsilon}{4} < \int_0^1 f\, dx \leq \sum m_j \Delta t_j$$
   we have
   $$\sum_{t_j \geq c} M_j' \Delta t_j - \frac{\epsilon}{4} < \int_0^1 f\, dx \leq \sum_{t_j \geq c} m_j' \Delta t_j.$$
   This also tells us that
   $$\left| \sum M_j \Delta t_j - \sum_{t_j \geq c} M_j' \Delta t_j \right| < \frac{\epsilon}{4}$$
   and
   $$\left| \sum m_j \Delta t_j - \sum_{t_j \geq c} m_j' \Delta t_j \right| < \frac{\epsilon}{4}$$
   so that
   $$\left| \int_0^1 f\, dx - \int_c^1 f\, dx \right| < \epsilon$$
   if $0 < c < \frac{\epsilon}{4}$.

(b) Consider $f(x) = (-1)^n(n+1)$.

□

3. (6.8) Supppose $f \in \mathcal{R}$ on $[a, b]$ for every $b > a$, where a is fixed. Define

$$\int_a^\infty f(x)\,dx = \lim_{x\to\infty} \int_a^b f(x)\,dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after f has been replaced by $|f|$, it is said to converge *absolutely*. Assume that $f \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^\infty f(x)\,dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges.

Proof. It is enough tocheck that

$$-f(0) + \sum_{k=1}^n f(k) \leq \int_1^n f(x)\,dx \leq \sum_{k=1}^{n-1} f(x).$$

Why is this enough?

□

Disclaimer: Some of these solutions have been taken from some outside sources, which will not be cited so that students do not find them. If you are interested in knowing where I got the solutions from please e-mail me.