Outlines

Homework 4

1. (6.6) Let P be the Cantor set. Let f be a bounded real function on $[0,1]$ which is continuous at every point outside P. Prove that $f \in \mathcal{R}$ on $[0,1]$.

   Proof. Let M be the supremum of $|f|$ on the interval. Cover P by a finite collection of open intervals such that $\sum b_i - a_i < \frac{\epsilon}{4M}$. Consider $\theta = \inf\{|x-y| : x \in Py \in [0,1] \setminus \bigcup(a_i,b_i)\}$. Let $E = \{x : d(x,P) \geq \frac{\theta}{2}\}$, this set is compact and f is uniformly continuous on E. We can then choose a $\delta > 0$ so that

   $$|f(x) - f(y)| < \frac{\epsilon}{2}$$

   for every $x, y \in E$ with $|x-y| < \delta$. Consider a partition such that $\max(t_i - t_{i-1}) < \min(\delta, \frac{\theta}{2})$. For this partition the difference of the upper and lower Riemann sums can be expressed as

   $$\sum (M_j - m_j)(t_j - t_{j-1}) = \Sigma_1 + \Sigma_2,$$

   where Σ_1 contains all the terms for which $[t_{j-1}, t_j]$ is contained in E and Σ_2 all the other terms. Notice

   $$\Sigma_1 \leq \frac{\epsilon}{2},$$

   and, since each interval $[t_{i-1}, t_i]$ that occurs in Σ_2 is contained in $\bigcup(a_i, b_i)$, we have

   $$\Sigma_2 < \frac{\epsilon}{2}.$$

   We conclude what we wanted. □

2. (6.10) Let p and q be positive real numbers such that

   $$\frac{1}{p} + \frac{1}{q} = 1.$$

   Prove the following statements.

   (a) If $u \geq 0$ and $v \geq 0$, then

   $$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

   Equality holds if and only if $u^p = v^q$.

   (b) If $f, g \in \mathcal{R}(\alpha)$ with $f, g \geq 0$, and

   $$\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha,$$

   then

   $$\int_a^b fg \, d\alpha \leq 1.$$

   (c) If f and g are complex functions in $\mathcal{R}(\alpha)$, then

   $$\left| \int_a^b fg \, d\alpha \right| \leq \left(\int_a^b |f|^p \, d\alpha \right)^{\frac{1}{p}} \left(\int_a^b |g|^q \, d\alpha \right)^{\frac{1}{q}}$$

   Proof.

(a) We will use as a fact that the exponential is convex, take the convex combination $\frac{\log(u^p)}{p} + \frac{\log(v^q}{q}$ and apply the definition of convex function on it. You get the desired inequality. To see that the equality holds if and only if $u^p = v^q$. For the equality consider the differentiable function $\phi(u) = \frac{u^p}{p} + \frac{v^q}{q} - uv$ and find its critical points.

(b) Integrate the latter inequality where $u = f$ and $v = g$.

(c) If either of the two integrals on the right is 0 we have nothing to prove (why is that?). Assume that both integrals on the right are positive. You can replace f by $\frac{|f(x)|}{\left(\int_a^b |f|^p \, d\alpha\right)^{\frac{1}{p}}}$ and g by $\frac{|g(x)|}{\left(\int_a^b |g|^p \, d\alpha\right)^{\frac{1}{p}}}$. These, together with the inequality $\left|\int \cdot d\alpha\right| \leq \int |\cdot| \, d\alpha$ and the last problem give you the desired inequality.

□

3. (6.11) Let α be fixed increasing function on $[a, b]$. For $u \in \mathcal{R}(\alpha)$, define

$$\|u\|_2 = \left(\int_a^b |u|^2 \, d\alpha\right)^{\frac{1}{2}}.$$

Suppose $f, g, h \in \mathcal{R}(\alpha)$, and prove the triangle inequality

$$\|f - g\|_2 \leq \|f - g\|_2 + \|g - h\|_2.$$

Proof.

$$\begin{aligned}
\|f - h\|_2 &= \int_a^b |f - h|^2 \, d\alpha \\
&= \int_a^b |f - g + g - h|^2 \, d\alpha \\
&\leq \int_a^b |f - g|^2 \, d\alpha + 2\int_a^b |f - g||g - h| \, d\alpha + \int_a^b |h - g|^2 \, d\alpha \\
&\leq \|f - g\|_2^2 + 2\|f - g\|_2\|h - g\|_2 + \|h - g\|_2^2 \\
&= \left(\|f - g\|_2 + \|h - g\|_2\right)^2.
\end{aligned}$$

□

4. (6.12) With the notations of the last problem, suppose $f \in \mathcal{R}(\alpha)$ and $\epsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $\|f - g\|_2 < \epsilon$.

Proof. Follow the hint on the book, g is piece-wise linear and so it is continuous. By definition of g (this is closely related to the fact that g is the weighted averages at the end points) we have that $|g - f|$ has as a maximum value $M_i - m_i$ on the interval $[x_{i-1}, x_i]$. You can choose the partition in such a way that

$$\sum (M_i - m_i)\left(\alpha(x_i) - \alpha(x_{i-1})\right) < \frac{\epsilon^2}{2M},$$

where M is the maximum of $|f|$ (why is f bounded?). This tells us that

$$\sum \left(M_i - m_i^2\right)\left(\alpha(x_i) - \alpha(x_{i-1})\right) < \epsilon^2$$

and so the upper sum of the Riemann integral of $|g - h|^2$ on this partition will also be less than ϵ^2. We have what we wanted. □

5. (6.13) Define

$$f(x) = \int_x^{x+1} \sin(t^2) \, dt.$$

(a) Prove that $|f(x)| < \frac{1}{x}$ if $x > 0$.

(b) Prove that

$$2xf(x) = \cos(x^2) - \cos\left((x + 1)^2\right) + r(x)$$

where $|r(x)| < \frac{c}{x}$ and c is a constant.

(c) Does $\int_0^\infty \sin(t^2)\,dt$ converge?

Proof.

(a) We only need to worry about the case where $x > 1$ (why?). Use integration by parts and notice that $f(x) < \frac{1}{x}$ (this is a straightforward computation after applying the inequality $|cos(x)| < 1$ to the remaining integral.) You can arrive to $f > \frac{-1}{x}$ with a similar argument (how?).

(b) Rewrite what you got in the last item after applying integration by parts to get

$$2xf(x) = \cos(x^2) - \cos\left((x+1)^2\right) + r(x)$$

and use the same approach from the last problem to bound r (what would $r(x)$ need to be? Is there an integral? Use integration by parts on that integral and notice that each of this terms are bounded by $\frac{1}{x^3}$. This will give you that, overall, $|r(x)| < \frac{c}{x}$ for some constant c).

(c) It does converge. We start by noticing that

$$\int_0^x \sin(t^2)\,dt = \int_0^{[x]} \sin(t^2)\,dt + \int_{[x]}^x \sin(t^2)\,dt = \sum_{i=0}^{[x]} \int_i^{i+1} \sin(t^2)\,dt + \int_{[x]}^x \sin(t^2)\,dt$$

where $[x]$ is the smallest integer lower than x. If we are able to check that the series $\sum \sum_{i=0}^{[x]} \int_i^{i+1} \sin(t^2)\,dt$ converges and that the limit of $\int_{[x]}^x \sin(t^2)\,dt$ as $x \to \infty$ both converge we are done (why can we do this?). To show that the series converge use part b and deduce that it does indeed converge (you need to do this). Finally, for the remaining integral we see that the integral goes to zero by applying integration by parts(why is this true?).

□

6. (6.15) Suppose f is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f^2\,dx = 1.$$

Prove that

$$\int_a^b xf(x)f'(x)\,dx = \frac{-1}{2}$$

and that

$$\int_a^b [f'(x)]^2\,dx \cdot \int_a^b x^2 f^2(x)\,dx > \frac{1}{4.}$$

Proof. The first part follows from integration by parts (how?). The second inequality follows from Problem 10c. □

7. (6.19) Let γ_1 be a curve in $\mathbb{R}^k$, defined on $[a, b]$; let ϕ be a continuous $1-1$ mapping of $[c, d]$ onto $[a, b]$, such that $\phi(c) = a$; and define $\gamma_2(s) = \gamma_1(\phi(s))$. Prove that γ_2 is an arc, a closed curve, or a rectifiable curve if and only if the same is true of γ_1. Prove that γ_2 and γ_1 have the same length.

Proof. We know $\gamma_1 = \gamma_2\,(\phi^{-1}(x))$ (Notice there is a reason why I want you to go over the equality) everywhere where ϕ^{-1} is well defined and continuous (why is this true?). From this you can deduce that they are both arcs, both closed curves, or both rectifiable curves (do it). Because both ϕ and its inverse are bijections there is a one to one correspondence between partitions, giving us that the two curves must have the same length. □

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