Activities

Féjer's theorem

For the following problems fill in the gaps on the proof, they are highlighted in blue. Any comment in brown shall be discussed with your classmates and resolved as a group.

1. Consider the *Dirichlet kernel*

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin\left(Nx + \frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}.$$

If

$$K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(x)$$

prove that

$$K_N(x) = \frac{1}{N+1}\frac{1-\cos(Nx+x)}{1-\cos x}$$

and that

(a) $K_N \geq 0$,

(b) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\,dx = 1$.

(c) $K_N(x) \leq \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}$ if $) < \delta \leq |x| \leq \pi$.

If $S_N = S_N(f;x)$ is the Nth partial sum of the Fourier series of f, consider the arithmetic means

$$\sigma_N = \frac{s_0 + \cdots s_N}{N+1}.$$

Prove that

$$\sigma_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\,dt,$$

and so prove Féjer's theorem: *If f is continuous, with period 2π, then $\sigma_N(f;x) \to f$ uniformly on $[-\pi,\pi]$.*

Proof. Use the identity $1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$ and the definition of D_n to show that

$$K_N(x) = \frac{1}{N+1}\frac{1-\cos(Nx+x)}{1-\cos x}.$$

(you may need another trigonometric identity)

(a) You can rewrite $K_N(x)$ as

$$\frac{1}{N+1}\left(\frac{\sin\left(\frac{N+1}{2}x\right)}{\sin(\frac{x}{2})}\right)^2.$$

The it is immediate that it is positive.

(b) Prove that $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)\,dx = 1$ and conclude what you wanted.

(c) Prove it.

Try to rewrite the partial sums of the Fourier series of f as a convolution of the Dirichlet kernel and f. Use that and the definition of K_N to get

$$\sigma_N(f;x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t)\, dt.$$

Show the uniform convergence by using (a), (b), and (c). □

2. Prove a pointwise version of Féjer's theorem: *If $f \in \mathcal{R}$ and $f(x+), f(x-)$ exist for some x then*

$$\lim_{N \to \infty} \sigma_N(f;x) = \frac{1}{2} \left(f(x+) + f(x-) \right).$$

Proof. □